Initialization of the Simplex Algorithm

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1 Abstract

The simplex method is an algorithm for solving linear problems which was discovered by George Dantzig in 1947. It occurs very frequently in almost every modern industry. In fact, areas using linear programming are as diverse as management, health, transportation, manufacturing, advertising, telecommunications, defense. Simplex helps to guide the management on the maximum or minimum investment in a particular portfolio. In general, the simplex method is an elegant solution to a common problem in planning and decision-making. In terms of widespread application, Dantzig's algorithm is one of the most successful algorithms of all time and ranked among the top 10 algorithms in the 20th century.

The purpose of a linear program (LP) is to maximize a linear objective function

$$z(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j = \langle \mathbf{c} | \mathbf{x} \rangle \tag{1.1}$$

subject to linear inequalities

$$\begin{gathered} \sum_{j=1}^{n} a_{ij} x_j = \langle \mathbf{a_i} | \mathbf{x} \rangle \leq b_i, \quad i = 1, \ldots, m \\ x_j \geq 0, \quad j = 1, \ldots, n. \end{gathered} \tag{1.2}$$

The vectors $\mathbf{c} = (c_1, \ldots, c_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$ denote the vector of coefficients in the goal function and the vector of unknown decision variables, respectively, and $\langle \mathbf{c} | \mathbf{x} \rangle$ denotes the scalar product of the vectors $\mathbf{c}$ and $\mathbf{x}$. The left hand side of any constraint in (1.2), given by

$$\langle \mathbf{a_i} | \mathbf{x} \rangle = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n, \; i = 1, \ldots, m,$$

denotes the scalar product of the vectors $\mathbf{a_i} = (a_{i1}, \ldots, a_{in})$ and $\mathbf{x}$.

The method of minimal angles (MA method, shortly) was introduced in [4] and it is aimed to solve linear optimization problems (1.1)-(1.2). The main idea used in this method arises from the graphical procedure for solving the linear programming problems.

It is known that in n-dimensional case any of the vertices of the polyhedron can be found by solving a system of n equations which are determined by some of the constraints (1.2). In [4] we propose a method for a proper selection of these equations, based on a generalization and formalization of the graphical procedure. The major idea guiding the MA method is the following: it is observable that the optimal vertex of the polytope is formed by the intersection of n constraints, where n is the number of variables included in the LP. These n constraints that form the optimal vertex should be the ones **closest in angle to the objective function**. In Figure 1 it is observable that the angles between the gradients $\mathbf{a_1}$ and $\mathbf{a_2}$ of the first and the second constraint and $\mathbf{c}$ are the smallest among the angles between all constraint gradients and $\mathbf{c}$.

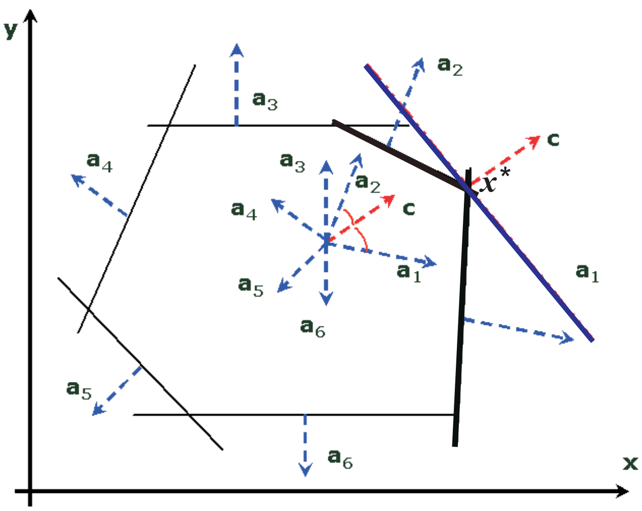

Figure 1: Illustration of the main idea of the minimal angles.

Algorithm 1.1. (MA method) *Suppose we have the linear maximization problem* (1.1)-(1.2) *with no redundant constraints. Let* $P \subseteq \mathbb{R}^n$ *be the set of feasible solutions defined by* (1.2). *Let* $\mathbf{c} = (c_1, \ldots, c_n)$ *be the gradient vector of the objective function and* $\mathbf{a_i} = (a_{i1}, \ldots, a_{in})$, $i = 1, \ldots, m$. *Consider the set*

$$V = \left\{ v_i = \frac{\langle \mathbf{c} | \mathbf{a_i} \rangle}{|\mathbf{a_i}|} = \cos(\mathbf{c}, \mathbf{a_i})|\mathbf{c}|, \ \ |\mathbf{r_i}| = \sqrt{\sum_{j=1}^{n} a_{ij}^2}, \quad i=1, \ldots, m \right\}. \tag{1.3}$$

Assume that the set V *contains* l *positive elements, denoted by* $v_{i_1}, \ldots, v_{i_l}$.

The following cases can be considered:

(a) *In the case* $l = 0$, *the maximal value of the objective function* $z(\mathbf{x})$ *is equal to* $z_{\max} = +\infty$.

(b) *In the case* $l \geq n$, *choose the initial iteration* $\mathbf{x_0}$ *of the simplex as the solution of the following system of equations:*

$$a_{i_k,1}x_1 + \cdots + a_{i_k,n}x_n = \langle \mathbf{a_{i_k}} | \mathbf{x} \rangle = b_{i_k}, \quad k = 1, \ldots, n, \tag{1.4}$$

where the indices $i_1, \ldots, i_n$ are corresponding to n maximal and positive values selected from the set V.

(c) *In the case $0 < l < n$, generate the following system of linear equations*

$$a_{i_k,1}x_1 + \cdots + a_{i_k,n}x_n = \langle \mathbf{a_{i_k}} | \mathbf{x} \rangle = b_{i_k}, \quad k = 1, \ldots, l, \tag{1.5}$$

wherein the indices $i_1, \ldots, i_l$ correspond to positive values $v_{i_1}, \ldots, v_{i_l}$ from the set V. Then, evaluate the basic solution $\mathbf{x_0}$ of the problem (1.1)-(1.2) by setting $n - l$ variables to zero and solving l equations in (1.5), which yields the remaining l variables, provided that these equations have a unique solution.

According to (1.3), it is observable that the constraint whose gradient generates a larger cosine value with the objective gradient is more likely to be included at an optimal extreme point than any with a smaller value.

When we started investigation on the idea of minimal angles, in 1998, it seemed that the idea of minimal angles will eliminate completely the necessity to use the simplex method to solve linear programs with positive coefficients without redundant constraints! In fact, after our research and after many comments on the MA method, the next reality was came out:
1. The output $\mathbf{x_0}$ of the MA method is a basic feasible solution of (1.1)-(1.2).
2. If the optimal solution in P is denoted by $\mathbf{x_P}$, then the following cases frequently occur:
(i) $\mathbf{x_0} = \mathbf{x_P}$, or
(ii) $\mathbf{x_0}$ and $\mathbf{x_P}$ belong on the same hyperplane of P.

In any case, the output $\mathbf{x_0}$ of the MA method could be used as the initial basic feasible solution of the simplex method.

Besides this important property, we mention the following useful property of the MA method. In the simplex method, all constraints together with added slack variables are used in each step. In the method of minimal angles, the number of active constraints is smaller with respect to the number of active constraints used in the corresponding simplex procedure. Moreover, slack variables are not used in the MA method. Therefore, dimensions of the problem considered in the MA method, are significantly smaller with respect to dimensions of the simplex procedure applied to the same problem. Hence, each substitution of a few iterations of the simplex method by only one application of the method of minimal angles usually significantly reduces the number of floating point operations and the spanned processor time.

As a method which is capable to create a good initial basis (initialization of the simplex), it has been followed and investigated in a number of papers or PhD theses.

Comments from [5]: "One of the methods, which the authors call the minimal angles method (MA method) was designed to determine either an optimal extreme point or an extreme point adjacent to an optimal extreme point."

Investigation in [1] extends the idea of minimal angles to the dual simplex method.

One characteristic comment is stated in [3]: "A different approach is taken by Stojković and Stanimirović [19], Junior and Lins [11], and Luh and Tsaih [13], who rather than improve on the simplex algorithm itself, developed a method to select a better starting point for the simplex method which reduces the number of simplex iterations needed."

The general conclusion from [2] is: "Although the method is very effective, and can create an optimal basis for linear programming on some occasions, it clearly can only create a good initial basis in general, not an exact optimal basis."

What is the general conclusion about the MA method? Clearly, the MA idea did not eliminate the need to use the simplex method to solve linear programs nor even the linear programs with positive coefficients and without redundant constraints! MA method will be remembered and used as one of good heuristics for the simplex method initialization. The most important conclusion is: there is no a magic wand to solve the LP problem in a single step.

References

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